A BIJECTIVE PROOF OF AN UNUSUAL SYMMETRIC GROUP GENERATING FUNCTION

MIKE ZABROCKI

Abstract. For $\sigma \in S_n$, let $D(\sigma) = \{i : \sigma_i > \sigma_{i+1}\}$ denote the descent set of σ. The length of the permutation is the number of inversions, denoted by $inv(\sigma) = \left|\{(i,j) : i < j, \sigma_i > \sigma_j\}\right|$. Define an unusual quadratic statisitic by $baj(\sigma) = \sum_{i \in D(\sigma)} i(n-i)$. We present here a bijective proof of the identity $\sum_{\substack{\sigma \in S_n \\ \sigma(n)=k}} q^{baj(\sigma)-inv(\sigma)} = \prod_{i=1}^{n-1} \frac{1-q^{i(n-i)}}{1-q^i}$ where k is a fixed integer.

The following identity was presented as a special case of a Weyl group generating function in a seminar talk at UCSD in November 1996 by John Stembridge. We present here a bijective proof of the identity.

Define a statistic on the permutations on n letters

$$baj(\sigma) = \sum_{i=1}^{n-1} i(n-i)\chi(\sigma_{i+1} < \sigma_i)$$

where χ is an indicator function

$$\chi(A) = \begin{cases} 1 \; if \;\; A \; true \\ 0 \; if \;\; A \; false \end{cases}$$

The number of inversions of a permutation may be expressed as

$$inv(\sigma) = \sum_{i=1}^{n-1} \sum_{j>i} \chi(\sigma_j < \sigma_i)$$

A special case of the main result presented in [2] when the root system is A_{n-1} is the following

Theorem 1.

$$\sum_{\sigma \in S_n} q^{baj(\sigma)-inv(\sigma)} = n \prod_{i=1}^{n-1} \frac{1-q^{i(n-i)}}{1-q^i}$$

A slightly stronger statement can be made for the specialization of this formula to this root system. For a fixed $k \in \{1 \ldots n\}$ the following equation also holds

Theorem 2.

$$\sum_{\substack{\sigma \in S_n \\ \sigma_n = k}} q^{baj(\sigma)-inv(\sigma)} = \prod_{i=1}^{n-1} \frac{1-q^{i(n-i)}}{1-q^i}$$

where k is an integer between 1 and n.

Notice that the right hand side of this equation can be expressed as a product of sums by the formula

$$\begin{aligned} \prod_{i=1}^{n-1} \frac{1-q^{i(n-i)}}{1-q^i} &= \prod_{i=1}^{n-1} \frac{1-q^{i(n-i)}}{1-q^{n-i}} = \prod_{i=1}^{n-1} \sum_{r_i=0}^{i-1} q^{(n-i)r_i} \\ &= \sum_{\substack{(r_1, r_2, \ldots, r_{n-1}) \\ 0 \le r_i < i}} q^{\sum_{i=1}^{n-1}(n-i)r_i} \end{aligned}$$

The object of this proof will be to find a bijection from the permutations, σ, of $\{1 \ldots n\}$ with $\sigma_n = k$ (k fixed) to sequences of integers $(r_1, r_2, \ldots r_{n-1})$ with the additional property that

$$baj(\sigma) - inv(\sigma) = \sum_{i=1}^{n-1} (n-i) r_i$$

Let σ be a permutation of $\{1 \ldots n\}$. σ can be represented by a sequence of integers $(v_1, v_2, \ldots, v_n)$ where v_i is $1 \le v_i \le i$ and is given by $v_i = \sum_{j \le i} \chi(\sigma_j \le \sigma_i)$.

Given such a sequence of v_i, it is possible to recover the permutation that it corresponds to by first constructing $\sigma' \in S_{n-1}$ for the sequence $(v_1, v_2, \ldots, v_{n-1})$ and then defining the permutation $\sigma \in S_n$ by $\sigma_n = v_n$, $\sigma_i = \sigma'_i$ if $\sigma'_i < v_n$, and $\sigma_i = \sigma'_i + 1$ if $\sigma_i \ge v_n$. This construction gives that the number of $i \in \{1 \ldots n\}$ such that $\sigma_i \le \sigma_n$ is v_n. This quantity does not change by building a larger permutation in the same manner.

Example 3. *Say* $v = (1, 1, 3, 1, 2, 5, 1)$

$v_1 = 1$ $\quad \sigma^{(1)} = 1$
$v_2 = 1$ $\quad \sigma^{(2)} = 21$
$v_3 = 3$ $\quad \sigma^{(3)} = 213$
$v_4 = 1$ $\quad \sigma^{(4)} = 3241$
$v_5 = 2$ $\quad \sigma^{(5)} = 43512$
$v_6 = 5$ $\quad \sigma^{(6)} = 436125$
$v_7 = 1$ $\quad \sigma^{(7)} = 5472361 = \sigma$

Notice that $\sigma_{i+1} < \sigma_i$ if and only if $v_{i+1} \le v_i$. This is because if $v_{i+1} \le v_i$ then $\sigma^{(i+1)}$ will have a descent in the i^{th} position, and this descent will remain for all $\sigma^{(k)}$ with $k > i$ (and in particular $\sigma^{(n)} = \sigma$).

Define the bijection from permutations σ with $\sigma_n = k$ by first computing the sequence of v_i and then setting $r_i = i\chi(v_{i+1} \leq v_i) + v_{i+1} - v_i - 1$.

This defines a map from such permutations to sequences of integers $(r_1, r_2, \ldots, r_{n-1})$ with $0 \leq r_i < i$. Note that if $v_i \geq v_{i+1}$ then $i - 1 \geq v_i - v_{i+1} \geq 0$ so that $0 \leq r_i = i - 1 - (v_i - v_{i+1}) \leq i - 1$. If $v_i < v_{i+1}$ then $0 < v_{i+1} - v_i \leq i$, hence $0 \leq r_i = v_{i+1} - v_i - 1 \leq i - 1$.

Given a sequence $(r_1, r_2, \ldots, r_{n-1})$ that is the image of some permutation and assume that the values of $v_{i+1}, \ldots, v_n$ are known, then $v_i - i\chi(v_{i+1} \leq v_i) = v_{i+1} - r_i + 1$. If the right hand side of the equation less than or equal to 0 then it must be that $\chi(v_{i+1} \leq v_i) = 1$ and so $v_i = i + v_{i+1} - r_i + 1$. Otherwise $\chi(v_{i+1} \leq v_i) = 0$ and then $v_i = v_{i+1} - r_i + 1$. The whole sequence of v_i can be recovered, and thus, the original permutation also.

This map is 1-1 since it is possible to recover the permutation if the sequence $(r_1, r_2, \ldots, r_{n-1})$ is given and the value of $v_n = k$ is known. There are the same number of permutations with the last element fixed as there are such sequences of numbers, hence this map is a bijection.

It remains to show the result

$$baj(\sigma) - inv(\sigma) = \sum_{i=1}^{n-1} (n-i) r_i$$

Note that v_i can be expressed by the formula $v_i = \sum_{j \leq i} \chi(\sigma_j \leq \sigma_i)$. Because $\binom{n+1}{2} = \sum_{i=1}^{n} \sum_{j \leq i} (\chi(\sigma_j \leq \sigma_i) + \chi(\sigma_j > \sigma_i)) = \sum_{i=1}^{n} v_i + inv(\sigma)$, the number of inversions of the permutation σ is given by the formula

$$inv(\sigma) = \binom{n+1}{2} - \sum_{i=1}^{n} v_i$$

The statistic baj can be given in terms of the v_i's because of the remark that $v_{i+1} \leq v_i$ if and only if $\sigma_{i+1} < \sigma_i$.

$$baj(\sigma) = \sum_{i=1}^{n-1} i(n-i)\chi(v_{i+1} \leq v_i)$$

Thus,

$$\begin{aligned}
\sum_{i=1}^{n-1} r_i(n-i) &= \sum_{i=1}^{n-1}(i\chi(v_{i+1} \leq v_i) + v_{i+1} - v_i - 1)(n-i) \\
&= \sum_{i=1}^{n-1} i(n-i)\chi(v_{i+1} \leq v_i) + \sum_{i=1}^{n-1} v_{i+1}(n-i) - \sum_{i=1}^{n-1}(v_i+1)(n-i) \\
&= \sum_{i=1}^{n-1} i(n-i)\chi(v_{i+1} \leq v_i) + \sum_{i=2}^{n} v_i(n-i+1) - \sum_{i=1}^{n-1} v_i(n-i) - \sum_{i=1}^{n-1} i \\
&= \sum_{i=1}^{n-1} i(n-i)\chi(v_{i+1} \leq v_i) + v_n + \sum_{i=2}^{n-1} v_i - v_1(n-1) - \binom{n}{2} \\
&= \sum_{i=1}^{n-1} i(n-i)\chi(v_{i+1} \leq v_i) + \sum_{i=1}^{n} v_i - v_1 n - \binom{n}{2} \\
&= \sum_{i=1}^{n-1} i(n-i)\chi(v_{i+1} \leq v_i) + \sum_{i=1}^{n} v_i - \binom{n+1}{2} \\
&= baj(\sigma) - inv(\sigma)
\end{aligned}$$

This shows the last property of the bijection and hence the theorem.

References

[1] N. Iwahori and H. Matsumato, On some Bruhat decomposition and the structure of the Hecke rings of p-adic Chevalley groups, *Inst. Hautes Études Sci. Publ. Math.* **25** (1965) 5–48.

[2] J. Stembridge and D. Waugh, A Weyl group generating function that ought to be better known, *Indag. Math.* 9 (1998), 451–457.

Mathematics and Statistics, TEL Building, York University, 4700 Keele Street, Toronto, Ontario, M3J 1P3, Canada